## Exercise 43

Two media with indices of refraction $n_{1}$ and $n_{2}$ are separated by a plane surface perpendicular to the unit vector $\mathbf{N}$. Let $\mathbf{a}$ and $\mathbf{b}$ be unit vectors along the incident and refracted rays, respectively, their directions being those of the light rays. Show that $n_{1}(\mathbf{N} \times \mathbf{a})=n_{2}(\mathbf{N} \times \mathbf{b})$ by using Snell's law, $\sin \theta_{1} / \sin \theta_{2}=n_{2} / n_{1}$, where $\theta_{1}$ and $\theta_{2}$ are the angles of incidence and refraction, respectively. (See Figure 1.3.11.)

figure 1.3.11 Snell's law.

## Solution

Start with Snell's law.

$$
\frac{\sin \theta_{1}}{\sin \theta_{2}}=\frac{n_{2}}{n_{1}}
$$

Multiply both sides by $n_{1} \sin \theta_{2}$.

$$
n_{1} \sin \theta_{1}=n_{2} \sin \theta_{2}
$$

Replace the argument of each sine function by $\pi$ minus the argument.

$$
n_{1} \sin \left(\pi-\theta_{1}\right)=n_{2} \sin \left(\pi-\theta_{2}\right)
$$

The point is that these new arguments represent the angles between $\mathbf{N}$ and the unit vectors as shown below.


Since $\|\mathbf{N}\|=\|\mathbf{a}\|=\|\mathbf{b}\|=1$, they can be placed on both sides.

$$
n_{1}\|\mathbf{N}\|\|\mathbf{a}\| \sin \left(\pi-\theta_{1}\right)=n_{2}\|\mathbf{N}\|\|\mathbf{b}\| \sin \left(\pi-\theta_{2}\right)
$$

Use the definition of the magnitude of the cross product.

$$
n_{1}\|\mathbf{N} \times \mathbf{a}\|=n_{2}\|\mathbf{N} \times \mathbf{b}\|
$$

Since the incident and refracted light rays both point downward, $\mathbf{N} \times \mathbf{a}$ and $\mathbf{N} \times \mathbf{b}$ will have the same direction. Therefore,

$$
n_{1}(\mathbf{N} \times \mathbf{a})=n_{2}(\mathbf{N} \times \mathbf{b}) .
$$

